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A CLASS OF DIMENSION-SKIPPING GRAPHS

Igor KŘÍŽ

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For $n \geqq 6$ there exists a graph G with $\dim G = n$, $\dim G^* \geqq n+2$, where G^* is G with a certain edge added.

The *dimension* $\dim G$ of a graph G (in this note a graph is a symmetric graph without loops) is the least number n such that G is embedable into $\prod_{i=1}^{n} G_i$, with G_i complete (see [1], [2], [3]). Equivalently, the dimension can be defined as follows (see e.g. [2], [3]): a set of edges of a graph is said to be an equivalence, if it constitutes of a disjoint system of cliques; the dimension of a graph G is the least number n such that the system of edges of the complement $c(G)$ of G can be written as $\bigcup_{i=1}^{n} E_i$, where the E_i are equivalences such that $\bigcap_{i=1}^{n} E_i = \emptyset$.

From the latter it is obvious that when removing an edge from a graph G, the dimension increases by at most one. On the other hand, it is not so obvious what happens when a single edge is added. One sess easily that the dimension doubles at worst and it is known that in fact it does not increase more than 3/2-times. There wasn't, however, known so far any example of an increase over $\dim G+1$. A construction of graphs where adding an edge causes an increase by two or more was formulated as a problem in [1]. In this note, we will present a class of graphs G_n ($n \geqq 6$) such that by adding a suitable edge one obtains a graph of dimension $\geqq \dim G_n + 2$.

Conventions and notation. A graph $G=(X, E)$ is a symmetric graph without loops. The edge connecting nodes x and y will be denoted by xy. A clique in G is the set of vertices of a complete subgraph of G. We will denote by $K(M)$ the set of edges of the complete graph with the set of vertices M. Thus, an equivalence relation mentioned above is a union $\bigcup_i K(M_i)$ with M_i disjoint. If $G=(X, E)$ is a graph, $c(G)$ designates the complement graph $(X, K(X) \setminus E)$. The cardinality of a set M will be denoted by $\#(M)$.

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The Construction. Let $x, y, a_1, \ldots, a_n$ be distinct points, $n \geqq 2$. Put $A = \{x, y, a_1, \ldots, \ldots, a_n\}$. Let sets $M_{i,j}$, $i, j = 1, \ldots, n$, $i \neq j$ be such that for any i, j, k, l, i', j'

1) $\#(M_{i,j} \cap M_{k,l}) \leqq 1$ if $i \neq k$ or $j \neq l$, $\#(M_{i,j} \cap M_{k,l}) = 1$ for $i \neq k$ and $j \neq l$.

2) $M_{i,j} \cap M_{k,l} \cap M_{i',j'} = \emptyset$ if the pairs (i, j), (k, l), (i', j') are distinct ($i = k = i'$ is excluded).

3) $$M_{i,j} \cap M_{k,j} = \emptyset \quad \text{if} \quad i \neq k.$$

4) $$M_{i,j} \cap M_{i,k} = \{a_i\} \quad \text{if} \quad j \neq k.$$

5) $$M_{i,j} \cap A = \{a_i\}.$$

6) $\#(M_{i,j})$ does not depend on i, j and exceeds n.

(e.g., we can set $M_{i,j} = \{a_i\} \cup \{\{(i, j), (k, l)\} | i \neq k,\ l \neq k,\ j \neq l,\ k, l = 1, \ldots, n\}$). Put $N_i = \{x, y, a_i\}$ for $i = 1, \ldots, n-1$ and $N_n = \{x, a_n\}$.

We construct a graph H_n as follows: the set of vertices is $\bigcup_{i,j} M_{i,j} \cup A$, the set of edges is $\bigcup_i K(N_i) \cup \bigcup_{i,j} K(M_{i,j})$. Further, put

$$G_n = c(H_n)$$

$$H_n^* = H_n \setminus \{xy\}$$

$$G_n^* = c(H_n^*).$$

(Thus, G_n^* is obtained from G_n by adding the edge xy.)

Theorem 1. *We have* $\dim G_n = n$.

Proof. Consider the equivalence relations $E_i = K(N_i) \cup \bigcup_j K(M_{j,i})$, $i = 1, \ldots, n$. According to the properties of $M_{i,j}$ and the definition of N_n we see easily that $\bigcap_{i=1}^{n} E_i = \emptyset$. Since the set of edges of H_n is equal to $\bigcup_{j=1}^{n} E_i$, $\dim G_n \leqq n$.

On the other hand, we have $\dim G_n \geqq n$: none of the n neighbours $a_1, \ldots, a_n$ of x are connected in H_n. ∎

Lemma 1. *Let the set* $B_i = \bigcup_j M_{i,j} \setminus \{a_i\}$ *be a union of* $n-1$ *cliques in* H_n, $n \geqq 2$. *Then these cliques coincide with* $M_{i,j} \setminus \{a_i\}$, $j = 1, \ldots, \hat{i}, \ldots, n$ *(the roof means omission).*

Proof. By 4), the sets $M_{i,j} \setminus \{a_i\}$ are disjoint. Hence, it suffices to show that for any other clique $C \subset B_i$ we have $\# C < \#(M_{i,j}) - 1$. Suppose the contrary. Then, obviously, the large C is not contained in any $M_{i,j}$. In consequence, we have $\# C(\cap \cap M_{i,j}) \leqq 1$: indeed, if not, we could choose distinct a, b, c in C such that $a \in M_{i,j}$, $b, c \in M_{i,j'}$ $(j \neq j')$. Since C is a clique, we have $ab \in K(M_{k,l})$ and $ac \in K(M_{k',l'})$ for some k, l, k', l'. By 2) necessarily $k = k'$, $l = l'$ so that $b, c \in M_{i,j'} \cap M_{k,l}$, contradicting 1).

But if $\#(C\cap M_{i,j})\leqq 1$ we conclude $\#(C)\leqq n-1$, while $\#(M_{i,j})>n$. ∎

Theorem 2. *For* $n\geqq 6$ *it holds*

$$\dim G_n^* \geqq n+2.$$

Proof. It will be done by contradiction. Let us suppose that we can cover H_n^* by $n+1$ equivalence relations $E_1, \ldots, E_{n+1}$. First, we prove that each of the $K(M_{i,j})$ $(i<n)$ is contained in some of the E_k. Let A_i be the set of all neighbours of a_i, put $A_{i,k}=\{u|a_iu\in E_k\}$. Clearly, $A_{i,k}$ is a clique in H^*. Thus, we have constructed a covering of A_i by $n+1$ cliques. Since A_i (as an induced subgraph of H_n) has three components, namely $\{x\}$, $\{y\}$ and B_i, we see that $n-1$ of the cliques have to cover B_i. By lemma 1, they coincide with the sets $M_{i,j}\setminus\{a_i\}$. Consequently, each of the $K(M_{i,j})$ is contained in an E_k.

Denote by $\bar{E}_k$ the union of all the $K(M_{i,j})$ with $i=1, \ldots, n-1$ and $j=1, \ldots, n$, contained in E_k. According to the properties of the $M_{i,j}$, each of the sets $\bar{E}_1, \ldots, \bar{E}_{n+1}$ has to be contained in some $P_i=\bigcup\{K(M_{j,i})|j=1, \ldots, i-1, i+1, \ldots, n-1\}$. (Note that P_n plays a special role). We can, hence, reindex the equivalences so that $\bar{E}_i\subset P_i$ for $i=1, \ldots, n$. As for $\bar{E}_{n+1}$ we have two possiblities:

Case 1. $\bar{E}_{n+1}\subset P_i$ for some $i<n$, say $\bar{E}_{n+1}\subset P_1$. Then $\bar{E}_2=P_2, \ldots, \bar{E}_n=P_n$ (see above), i.e. the vertex a_1 meets all the relations $\bar{E}_2, \ldots, \bar{E}_n$. Then a_1x, a_1y belong to E_1, E_{n+1}. This implies that at most two of the other a_jx, a_ky belong to $E_1\cup E_{n+1}$ and hence (recall that $n\geqq 6$) $a_ix, a_iy\in E_2\cup\ldots\cup E_n$ for some $i\in\{2, \ldots, n-1\}$. But there is only one of the $E_2, \ldots, E_n$ left for this purpose, namely E_i, all the others contain some a_iu with $u\in B_i$.

Case 2. $\bar{E}_{n+1}\subset P_n$. Then $\bar{E}_2=P_1, \ldots, \bar{E}_{n-1}=P_{n-1}$. Now for at most four indices we can have either $a_ix\in E_n\cup E_{n+1}$ or $a_iy\in E_n\cup E_{n+1}$. Thus, for some $j\in\{1, \ldots, n-1\}$ we have (recall $n\geqq 6$) $a_jx, a_jy\in E_1, \ldots, E_{n-1}$. Again, we have got a contradiction, since only E_j is left for use. ∎

Remark. For $n<6$ our construction does not work; we have there $\dim G_n^*=n+1$. For $n=6$ $\dim G_n=8$. We do not know whether $\dim G_n^*=n+2$ holds generally for $n\geqq 6$, or if perhaps some of the G_n^* skip more; a general upper estimate seems to be $n+4$.

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Igor Kříž

Matematicko-fyzikálni Fakulta
Univerzity Karlovy
18600 Praha 8
Sokolovská 83
ČSSR